International Journal of Theoretical Physics, Vol. 6, No. 2 (1972), p. 155–164

Non-Relativistic Approximation of the Theory of the Particle of Spin Maximum 1

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Received: 29 *July* 1971

1. *The Non-Relativistic Equations of the Particle of Spin Maximum* 1 *and Charge q in an Electromagnetic Field*

In a previous paper (Pereira, 1972) we introduced the equations describing the particle of charge q and spin maximum 1 moving in an electromagnetic field given by the potentials $\mathbf{A}$ and V, namely

$$\left[\frac{\epsilon}{c}\frac{a_4 + b_4}{2} + \sum_k^3 \pi_k \frac{a_4 b_k + a_k b_4}{2} - m_0 c a_4 b_4\right]_{\text{op}} \psi = 0 \tag{1.1}$$

in which ψ is a column matrix with sixteen elements ψ_{ik},

$$\psi_{11}\,\psi_{12}\,\psi_{13}\,\psi_{14}\,\psi_{21}\,\psi_{22}\,\psi_{23},\ldots,\psi_{42}\,\psi_{43}\,\psi_{44},$$

whereas ϵ_{op} and $\boldsymbol{\pi}_{\text{op}}$ denote the operators

$$\epsilon_{\text{op}} = (E - qV)_{\text{op}} = -i\hbar\,\partial_t - qV, \qquad \boldsymbol{\pi}_{\text{op}} = \left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)_{\text{op}} = i\hbar\nabla - \frac{q}{c}\mathbf{A} \tag{1.2}$$

As to the a_μ, b_μ, they are the 16×16 matrices

$$a_\mu = \alpha_\mu \times I, \qquad b_\mu = I \times \alpha_\mu \tag{1.3}$$

built with the 4×4 identity matrix I and the Dirac matrices α_μ, whose values we shall henceforth fix as follows

$$\alpha_1 = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, \qquad \alpha_2 = \begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{vmatrix}$$

$$\alpha_3 = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \qquad \alpha_4 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \tag{1.4}$$

The system (1.1) can, however, be put in the equivalent form (see Pereira, 1971)

$$\left(\frac{\epsilon}{c} - m_0 c\right)\chi_1 + \tfrac{1}{2}(-\theta_2 \chi_2 - \theta_1 \chi_3) = 0 \tag{1.5a}$$

$$\left(-\frac{\epsilon}{c} - m_0 c\right)\chi_4 + \tfrac{1}{2}(\theta_1 \chi_2 + \theta_2 \chi_3) = 0 \tag{1.5b}$$

$$m_0 c \chi_2 + \tfrac{1}{2}(-\theta_2 \chi_1 + \theta_1 \chi_4) = 0 \tag{1.5c}$$

$$m_0 c \chi_3 + \tfrac{1}{2}(-\theta_1 \chi_1 + \theta_2 \chi_4) = 0 \tag{1.5d}$$

which proves to be more adequate than (1.1) in order to calculate its non-relativistic approximation, and where $\chi_1, \chi_2, \chi_3, \chi_4$ are the column matrices

$$\chi_1 = \begin{vmatrix} \psi_{11} \\ \psi_{12} \\ \psi_{21} \\ \psi_{22} \end{vmatrix}, \quad \chi_2 = \begin{vmatrix} \psi_{13} \\ \psi_{14} \\ \psi_{23} \\ \psi_{24} \end{vmatrix}, \quad \chi_3 = \begin{vmatrix} \psi_{31} \\ \psi_{32} \\ \psi_{41} \\ \psi_{42} \end{vmatrix}, \quad \chi_4 = \begin{vmatrix} \psi_{33} \\ \psi_{34} \\ \psi_{43} \\ \psi_{44} \end{vmatrix} \tag{1.6}$$

θ_1 and θ_2 denoting the 4×4 matrices

$$\theta_1 = \begin{vmatrix} \pi_z & 0 & \pi_x - i\pi_y & 0 \\ 0 & \pi_z & 0 & \pi_x - i\pi_y \\ \pi_x + i\pi_y & 0 & -\pi_z & 0 \\ 0 & \pi_x + i\pi_y & 0 & -\pi_z \end{vmatrix},$$

$$\theta_2 = \begin{vmatrix} \pi_x & \pi_x - i\pi_y & 0 & 0 \\ \pi_x + i\pi_y & -\pi_z & 0 & 0 \\ 0 & 0 & \pi_z & \pi_x - i\pi_y \\ 0 & 0 & \pi_x + i\pi_y & -\pi_z \end{vmatrix} \tag{1.7}$$

From (1.5c) and (1.5d) one gets

$$\begin{aligned} \chi_2 &= (2m_0 c)^{-1}(\theta_2 \chi_1 - \theta_1 \chi_4) \\ \chi_3 &= (2m_0 c)^{-1}(\theta_1 \chi_1 - \theta_2 \chi_4) \end{aligned} \tag{1.8}$$

so that (1.5a) and (1.5b) become

$$\left[\frac{\epsilon}{c} - m_0 c - (4m_0 c)^{-1}(\theta_1{}^2 + \theta_2{}^2)\right]\chi_1 = -(4m_0 c)^{-1}(\theta_1 \theta_2 + \theta_2 \theta_1)\chi_4 \tag{1.9a}$$

$$\left[\frac{\epsilon}{c} + m_0 c + (4m_0 c)^{-1}(\theta_1{}^2 + \theta_2{}^2)\right]\chi_4 = (4m_0 c)^{-1}(\theta_1 \theta_2 + \theta_2 \theta_1)\chi_1 \tag{1.9b}$$

Equation (1.9b) now provides the expression of χ_4 as a function of χ_1, and by introducing it in (1.9a) we then obtain the equivalent form of (1.5a)–(1.5d),

$$\left\{\epsilon - m_0 c^2 - \frac{\theta_1{}^2 + \theta_2{}^2}{4m_0} + \frac{\theta_1\theta_2 + \theta_2\theta_1}{16 m_0{}^2 c} \times \left[\frac{\epsilon}{c} + m_0 c + \frac{\theta_1{}^2 + \theta_2{}^2}{4m_0 c}\right]^{-1} (\theta_1\theta_2 + \theta_2\theta_1)\right\}\chi_1 = 0 \quad (1.10a)$$

$$\chi_4 = \left[\frac{\epsilon}{c} + m_0 c + \frac{\theta_1{}^2 + \theta_2{}^2}{4m_0 c}\right]^{-1} \frac{\theta_1\theta_2 + \theta_2\theta_1}{4m_0 c}\chi_1 \quad (1.10b)$$

$$\chi_2 = (2m_0 c)^{-1}(-\theta_1\chi_4 + \theta_2\chi_1) \quad (1.10c)$$

$$\chi_3 = (2m_0 c)^{-1}(-\theta_2\chi_4 + \theta_1\chi_1) \quad (1.10d)$$

whose non-relativistic approximation we shall now calculate.

As it will be discussed in another paper (Pereira, 1971), the non-relativistic theories can be obtained from the corresponding relativistic theories by carrying on a 'β-approximation', which means that in the developments in series of β^n ($\beta = v/c$) one must retain only the terms up to the order of β^1 and neglect the others. According to this, the β-approximation of the operator $(\epsilon/c) - m_0 c$ then becomes†

$$\frac{\epsilon_r}{c} - m_0 c \cong \frac{1}{c}(E_n - qV) = \frac{\epsilon_n}{c},$$

whereas for

$$\left[\frac{\epsilon}{c} + m_0 c + \frac{\theta_1{}^2 + \theta_2{}^2}{4m_0 c}\right]^{-1}$$

we get

$$\left[\frac{\epsilon}{c} + m_0 c + \frac{\theta_1{}^2 + \theta_2{}^2}{4m_0 c}\right]^{-1} = (2m_0 c)^{-1}\left[1 - \frac{E_n - qV}{2m_0 c^2} - \frac{\theta_1{}^2 + \theta_2{}^2}{8m_0{}^2 c^2} + \dots\right]$$

Let us return to (1.7) and, by making use of the Pauli matrices,

$$\sigma_1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \qquad \sigma_2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \qquad \sigma_3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \quad (1.11)$$

write

$$\theta_1 = (\boldsymbol{\pi}.\boldsymbol{\sigma}) \times I, \qquad \theta_2 = I \times (\boldsymbol{\pi}.\boldsymbol{\sigma}) \quad (1.12)$$

The operator $\theta_1{}^2 + \theta_2{}^2$ then takes the form

$$\theta_1{}^2 + \theta_2{}^2 = [(\boldsymbol{\pi}.\boldsymbol{\sigma})^2 \times I] + [I \times (\boldsymbol{\pi}.\boldsymbol{\sigma})^2]$$
$$= \left[\left(\boldsymbol{\pi}^2 + \frac{\hbar q}{c}\boldsymbol{\sigma}.\mathbf{H}\right) \times I\right] + \left[I \times \left(\boldsymbol{\pi}^2 + \frac{\hbar q}{c}\boldsymbol{\sigma}.\mathbf{H}\right)\right] \quad (1.13)$$

† From now on, we shall often make use of the subscripts r or n in order to denote that a given physical magnitude is taken either relativistically or non-relativistically.

where $\mathbf{H} = \mathrm{rot}\,\mathbf{A}$ is the magnetic field. It follows from the preceding that the term $\theta_1{}^2 + \theta^2/(8m_0{}^2c^2)$ is of the order of β^2, and must therefore be neglected.

$$\left[\frac{\epsilon}{c} + m_0 c + \frac{\theta_1{}^2 + \theta_2{}^2}{4m_0 c}\right]^{-1} \cong \frac{1}{2m_0 c} \tag{1.14}$$

Furthermore, $\theta_1\theta_2 + \theta_2\theta_1/(8m_0{}^2c^2)$ can be written

$$\frac{\theta_1\theta_2 + \theta_2\theta_1}{8m_0{}^2c^2} = (8m_0{}^2c^2)^{-1}\{[(\boldsymbol{\pi}.\boldsymbol{\sigma}) \times (\boldsymbol{\pi}.\boldsymbol{\sigma})] + [(\boldsymbol{\pi}.\boldsymbol{\sigma}) \times (\boldsymbol{\pi}.\boldsymbol{\sigma})]\}$$
$$= (4m_0{}^2c^2)^{-1}\left\{\left[\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right).\boldsymbol{\sigma}\right] \times \left[\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right).\boldsymbol{\sigma}\right]\right\}$$

which means that it is of the order of β^2, or, more precisely, that equation (1.10b) expresses the fact that χ_4 is of the order of β^2 with regard to χ_1. Thence

$$\chi_4 \cong 0 \tag{1.15}$$

so that one is led to the simplified form of equations (1.10c) and (1.10d),

$$\chi_2 \cong (2m_0 c)^{-1}\theta_2\chi_1, \qquad \chi_3 \cong (2m_0 c)^{-1}\theta_1\chi_1 \tag{1.16}$$

According to the preceding results, we finally get the non-relativistic approximation of (1.10a)–(1.10d),

$$\left\{\epsilon - \frac{\boldsymbol{\pi}^2}{2m_0} - \frac{\hbar q}{2m_0 c}\left[\frac{(\boldsymbol{\sigma} \times I) + (I \times \boldsymbol{\sigma})}{2}\right].\mathbf{H}\right\}_{\mathrm{op}}\chi_1 = 0 \tag{1.17}$$
$$\chi_2 = \frac{\theta_2}{2m_0 c}\chi_1$$
$$\chi_3 = \frac{\theta_1}{2m_0 c}\chi_1$$
$$\chi_4 = 0$$

that is, by making use of the explicit form of the operators ϵ, $\boldsymbol{\pi}$, θ_1, θ_2 as defined in (1.2) and (1.12),

$$\left[i\hbar\,\partial_t + qV + \frac{1}{2m_0}\left(i\hbar\nabla - \frac{q}{c}\mathbf{A}\right)^2 + \frac{\hbar q}{2m_0 c}\frac{(\boldsymbol{\sigma} \times I) + (I \times \boldsymbol{\sigma})}{2}.\mathbf{H}\right]\begin{bmatrix}\psi_{11}\\ \psi_{12}\\ \psi_{21}\\ \psi_{22}\end{bmatrix} = 0 \tag{1.18a}$$

$$\begin{bmatrix}\psi_{13}\\ \psi_{14}\\ \psi_{23}\\ \psi_{24}\end{bmatrix} = (2m_0 c)^{-1}\left(I \times \begin{bmatrix} i\hbar\,\partial_z - \frac{q}{c}A_z & i\hbar(\partial_x - i\,\partial_y) - \frac{q}{c}(A_n - iA_y) \\ i\hbar(\partial_x + i\,\partial_y) - \frac{q}{c}(A_x + iA_y) & -i\hbar\,\partial_z + \frac{q}{c}A_z \end{bmatrix}\right)\begin{bmatrix}\psi_{11}\\ \psi_{12}\\ \psi_{21}\\ \psi_{22}\end{bmatrix} \tag{1.18b}$$

$$\begin{bmatrix}\psi_{31}\\ \psi_{32}\\ \psi_{41}\\ \psi_{42}\end{bmatrix} = (2m_0 c)^{-1}\left(\begin{bmatrix} i\hbar\,\partial_z - \frac{q}{c}A_z & i\hbar(\partial_x - i\,\partial_y) - \frac{q}{c}(A_x - iA_y) \\ i\hbar(\partial_x + i\,\partial_y) - \frac{q}{c}(A_x + iA_y) & -i\hbar\,\partial_z + \frac{q}{c}A_z \end{bmatrix} \times I\right)\begin{bmatrix}\psi_{11}\\ \psi_{12}\\ \psi_{21}\\ \psi_{22}\end{bmatrix} \tag{1.18c}$$

$$\psi_{33} = \psi_{34} = \psi_{43} = \psi_{44} = 0 \tag{1.18d}$$

Let us now recall to mind the formal procedure which, starting from the generalised Dirac equations, leads to the generalised equations of the particle of spin maximum 1 (Pereira, 1971a). [This procedure is just a particular application of the so-called 'method of the fusion' (see de Broglie, 1954)]. It can actually be said that in order to obtain the later equations, one must but replace the matrices $M = \alpha_1, \alpha_2, \alpha_3, \alpha_4, I$ appearing in the equation of Dirac by the matrices

$$\frac{(M \times I)(I \times \alpha_4) + (I \times M)(\alpha_4 \times I)}{2}$$

and at the same time substitute the wave function with four components by a wave function with $4 \times 4 = 16$ components. Now, it follows from (1.18a) that the same formal procedure is still valid at the non-relativistic level, for if one substitutes in the equation of Pauli

$$\left[i\hbar\,\partial_t + qV + (2m_0)^{-1}\left(i\hbar\nabla - \frac{q}{c}\mathbf{A}\right)^2 + \frac{\hbar q}{2m_0 c}\boldsymbol{\sigma}.\mathbf{H}\right]\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} = 0$$

the matrices

$$M = \sigma_1, \sigma_2, \sigma_3, I \equiv \sigma_4 \tag{1.19}$$

by

$$\frac{(M \times I)(I \times \sigma_4) + (I \times M)(\sigma_4 \times I)}{2} \tag{1.20}$$

and let a wave function with $2 \times 2 = 4$ components take the place of the wave function with two components, equation (1.18a) is then obtained.

We may accordingly assert that the method of the fusion still remains true within the frame of non-relativistic wave mechanics, since it provides the equations describing the fusion of two corpuscles of Pauli, i.e., the non-relativistic equations of evolution (1.18a) of the particle with spin maximum 1 and charge q.

2. *The Non-Relativistic Formalism*

It can easily be verified that equations (1.18a)–(1.18c) bring about an equation of continuity, that is, a relation of the form

$$\partial_t \rho + \operatorname{div} \mathbf{j} = 0 \tag{2.1}$$

where ρ and $\mathbf{j}$ are given by

$$\rho_n = \chi_1{}^* \chi_1 \tag{2.2}$$

$$\mathbf{j}_n = \frac{i\hbar}{2m_0}(\chi_1{}^* \nabla \chi_1 - \chi_1 \nabla \chi_1{}^*) - \frac{q}{2m_0 c} \chi_1{}^* \chi_1 \mathbf{A} \tag{2.3}$$

These expressions which we shall assume as providing the probability density and the probability flow vector, can be deduced as well by performing the non-relativistic approximation directly upon the relativistic definitions of ρ and $\mathbf{j}$ arising from (1.1) (see Pereira, 1972)

$$\rho_r = \psi^* \frac{a_4 + b_4}{2} \psi \tag{2.4}$$

$$\mathbf{j}_r = -c\psi^* \frac{a_4 \vec{\mathbf{b}} + \vec{\mathbf{a}} b_4}{2} \psi \tag{2.5}$$

For us to make sure, let us start with ρ_r and, by making use of (1.3), (1.4) and (1.15), write

$$\begin{aligned} \rho_r &= \psi_{11}^* \psi_{11} + \psi_{12}^* \psi_{12} + \psi_{21}^* \psi_{21} + \psi_{22}^* \psi_{22} - \psi_{33}^* \psi_{33} - \psi_{34}^* \psi_{34} \\ &\quad - \psi_{43}^* \psi_{43} - \psi_{44}^* \psi_{44} \\ &\cong \psi_{11}^* \psi_{11} + \psi_{12}^* \psi_{12} + \psi_{21}^* \psi_{21} + \psi_{22}^* \psi_{22} = \chi_1{}^* \chi_1 \end{aligned}$$

so that we get the non-relativistic expression (2.2).

Let us now consider the vector $\mathbf{j}_r$. For the sake of similarity, we shall only calculate the non-relativistic approximation of its first component,

$$\mathbf{j}_{r_1} = -c\psi^* \frac{a_4 b_1 + a_1 b_4}{2} \psi$$

the reasoning being in all points identical for j_{r_2} and j_{r_3}. It then follows from (1.3) and (1.4) that

$$\begin{aligned} \frac{2}{c} j_{r_1} &\cong \psi_{11}^*(\psi_{14} + \psi_{41}) + \psi_{12}^*(\psi_{13} + \psi_{42}) + \psi_{21}^*(\psi_{24} + \psi_{31}) \\ &\quad + \psi_{22}^*(\psi_{23} + \psi_{32}) + \text{conj} \\ &= [\psi_{11}^* \psi_{12}^* \psi_{21}^* \psi_{22}^*] \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_{13} \\ \psi_{14} \\ \psi_{23} \\ \psi_{24} \end{bmatrix} \\ &\quad + [\psi_{11}^* \psi_{12}^* \psi_{21}^* \psi_{22}^*] \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_{31} \\ \psi_{32} \\ \psi_{41} \\ \psi_{42} \end{bmatrix} + \text{conj} \end{aligned}$$

that is, according to (1.6) and (1.11),

$$\frac{2}{c}j_{r_1} \cong \chi_1{}^*(I \times \sigma_1)\chi_2 + \chi_1{}^*(\sigma_1 \times I)\chi_3 + \text{conj}$$

If we now replace χ_2 and χ_3 by their non-relativistic expressions (1.18b) and (1.18c), we are led to the relations

$$4m_0 j_{r_1} \cong \chi_1{}^*(I \times \sigma_1)(I \times \boldsymbol{\pi}.\boldsymbol{\sigma})\chi_1 + \chi_1{}^*(\sigma_1 \times I)(\boldsymbol{\pi}.\boldsymbol{\sigma} \times I)\chi_1 + \text{conj}$$

that is, by means of (1.2) and (1.11),

$$\begin{aligned} 4m_0 j_{r_1} \cong\ & 2i\hbar(\psi_{11}^* \partial_1 \psi_{11} + \psi_{12}^* \partial_1 \psi_{12} + \psi_{21}^* \partial_1 \psi_{21} + \psi_{22}^* \partial_1 \psi_{22} - \text{conj}) \\ & - \frac{4q}{c} A_1(\psi_{11}^* \psi_{11} + \psi_{12}^* \psi_{12} + \psi_{21}^* \psi_{21} + \psi_{22}^* \psi_{22}) \\ & - \hbar\{\psi_{11}^*[2\,\partial_2 \psi_{11} + i\,\partial_z(\psi_{12} + \psi_{21})] \\ & + i\psi_{12}^* \partial_z(-\psi_{11} + \psi_{22}) + i\psi_{21}^* \partial_z(-\psi_{11} + \psi_{22}) \\ & + \psi_{22}^*[-i\,\partial_z(\psi_{12} + \psi_{21}) - 2\,\partial_2 \psi_{22}] + \text{conj}\} \end{aligned}$$

This expression can still be written

$$\begin{aligned} 4m_0 j_{r_1} \cong\ & (2i\hbar\chi_1{}^* \nabla_1 \chi_1 + \text{conj}) - \frac{4q}{c} A_1 \chi_1{}^* \chi_1 \\ & - \hbar\{\chi_1{}^*[(I \times (\boldsymbol{\nabla} \wedge \boldsymbol{\sigma})_1) + ((\boldsymbol{\nabla} \wedge \boldsymbol{\sigma})_1 \times I)]\chi_1 + \text{conj}\} \end{aligned}$$

where $\boldsymbol{\nabla} \wedge \boldsymbol{\sigma}$ is the 3-component matrix operator

$$(\boldsymbol{\nabla} \wedge \boldsymbol{\sigma})_1 = \partial_2 \sigma_3 - \partial_3 \sigma_2, \qquad (\boldsymbol{\nabla} \wedge \boldsymbol{\sigma})_2 = \partial_3 \sigma_1 - \partial_1 \sigma_3,$$
$$(\boldsymbol{\nabla} \wedge \boldsymbol{\sigma})_3 = \partial_1 \sigma_2 - \partial_2 \sigma_1$$

We then arrive at the final form of the non-relativistic approximation of the probability flow vector $\mathbf{j}_r$,

$$\mathbf{j}_r \cong \frac{i\hbar}{2m_0}(\chi_1{}^* \boldsymbol{\nabla}\chi_1 - \chi_1 \boldsymbol{\nabla}\chi_1{}^*) - \frac{q}{m_0 c}\mathbf{A}\chi_1{}^* \chi_1 + \mathbf{u} \tag{2.6}$$

where

$$\mathbf{u} = \frac{-\hbar}{2m_0}\chi_1{}^* \frac{(I \times (\boldsymbol{\nabla} \wedge \boldsymbol{\sigma})) + ((\boldsymbol{\nabla} \wedge \boldsymbol{\sigma}) \times I)}{2}\chi_1 + \text{conj} \tag{2.7}$$

As it arises from the comparison of (2.3) and (2.6), the non-relativistic approximation of $\mathbf{j}_r$ does not lead to the definition of $\mathbf{j}_n$, but rather differs from it in a vector $\mathbf{u}$ given by (2.7). Yet the agreement of both expressions can be reached if $\boldsymbol{\nabla}.\mathbf{u} = 0$. As a matter of fact, one must realise that in wave mechanics what is closely related to the equations of evolution is but the equation of continuity (2.1), not the definitions of $\mathbf{j}$ and ρ, which involve some arbitrariness. Let us suppose, for instance, that we take, instead of $\mathbf{j}$, the vector $\mathbf{j} + \mathbf{v}$ with $\boldsymbol{\nabla}.\mathbf{v} = 0$. It is then manifest that in both cases we are led to the same equation of continuity (2.1), so that we may assert that a vector $\mathbf{v}$ with null divergence must be taken as physically meaningless in the definition of $\mathbf{j}$. Now, this is what actually happens with $\mathbf{j}_r \cong \mathbf{j}_n + \mathbf{u}$, for it can easily be seen that $\nabla . \mathbf{u} = 0$ (we shall omit this calculation for it presents no difficulty). Moreover, one must emphasize the fact that the same anomaly occurs when dealing with the probability flow vector of the Dirac theory, $\mathbf{j}_D = -c\psi^* \boldsymbol{\alpha}\psi$, whose non-relativistic approximation leads to

$$\mathbf{j}_D \cong \frac{i\hbar}{2m_0}(\chi_1{}^* \nabla\chi_1 - \chi_1 \nabla\chi_1{}^*) - \frac{q}{2m_0 c}\mathbf{A}\chi_1{}^* \chi_1 + \mathbf{u}_D$$

which differs in a vector

$$\mathbf{u}_D = \frac{-\hbar}{2m_0}\chi_1{}^*(\nabla \wedge \boldsymbol{\sigma})\chi_1 + \text{conj}$$

from the well-known expression given in the theory of Pauli (Pereira, 1971). However, it can easily be verified that we have again $\nabla . \mathbf{u}_D = 0$. Besides, the comparing of $\mathbf{u}$ and $\mathbf{u}_D$ again confirms the procedure of the fusion we have referred to, inasmuch as $\mathbf{u}$ can be formally obtained from $\mathbf{u}_D$ by replacing the matrices $\nabla \wedge \boldsymbol{\sigma}$ by

$$\frac{(I \times (\nabla \wedge \boldsymbol{\sigma})) + ((\nabla \wedge \boldsymbol{\sigma}) \times I)}{2}$$

while the 4-component wave function of Dirac takes the place of the 2-component wave function of Pauli.

In order to render more complete the study of the non-relativistic formalism of the theory, let us next calculate the approximation of the relativistic Lagrangian of the particle with spin maximum 1 and charge q moving in a field given by the electromagnetic potentials $\mathbf{A}$ and V, which can be written (Pereira, 1971)

$$\begin{aligned}\mathscr{L} = \frac{c}{2}\Big\{&\chi_1{}^*\left(\frac{\epsilon}{c} - m_0 c\right)\chi_1 + \tfrac{1}{2}\chi_1{}^*(-\theta_2\chi_2 - \theta_1\chi_3) + \chi_4{}^*\left(-\frac{\epsilon}{c} - m_0 c\right)\chi_4 \\ &+ \tfrac{1}{2}\chi_4{}^*(\theta_1\chi_2 + \theta_2\chi_3) + m_0 c\chi_2{}^*\chi_2 + \tfrac{1}{2}\chi_2{}^*(-\theta_2\chi_1 + \theta_1\chi_4) \\ &+ m_0 c\chi_3{}^*\chi_3 + \tfrac{1}{2}\chi_3{}^*(-\theta_1\chi_1 + \theta_2\chi_4)\Big\} + \text{conj}\end{aligned}$$

with χ_μ and θ_k given by (1.6) and (1.12). Now at the non-relativistic approximation we have

$$\frac{\epsilon_r}{c} - m_0 c \cong \frac{\epsilon_n}{c}$$

and

$$-\frac{\epsilon_r}{c} - m_0 c \cong -\frac{\epsilon_n}{c} - 2m_0 c$$

so that $\mathscr{L}$ becomes

$$\begin{aligned}\mathscr{L} = \frac{c}{2}\Big\{&\chi_1{}^*\frac{\epsilon}{c}\chi_1 + \tfrac{1}{2}\chi_1{}^*(-\theta_2\chi_2 - \theta_1\chi_3) + m_0 c(\chi_2{}^*\chi_2 + \chi_3{}^*\chi_3) \\ &- \tfrac{1}{2}(\chi_2{}^*\theta_2\chi_1 + \chi_3{}^*\theta_1\chi_1) + \tfrac{1}{2}(\chi_2{}^*\theta_1\chi_4 + \chi_3{}^*\theta_2\chi_4) \\ &- \chi_4{}^*\frac{\epsilon}{c}\chi_4 - 2m_0 c\chi_4{}^*\chi_4 + \tfrac{1}{2}\chi_4{}^*(\theta_1\chi_2 + \theta_2\chi_3)\Big\} + \text{conj}\end{aligned}$$

Since, as has been stated above, the transition to the non-relativistic level is provided by means of a β-approximation, one must neglect in the preceding expression the last four terms in the bracket, for they are of the order of β^2 with regard to χ_1. The Lagrangian then takes the form

$$\mathscr{L} \cong \mathscr{L}_n \equiv \chi_1^* \frac{\epsilon}{2} \chi_1 + \frac{m_0 c^2}{2}(\chi_2^* \chi_2 + \chi_3^* \chi_3)$$
$$- \frac{c}{4}(\chi_1^* \theta_2 \chi_2 + \chi_1^* \theta_1 \chi_3 + \chi_2^* \theta_2 \chi_1 + \chi_3^* \theta_1 \chi_1) + \text{conj}$$

where ϵ and $\boldsymbol{\pi}$ are the energy and momentum operators defined in (1.2).

Let us now make use of the method of the fusion in order to obtain the non-relativistic Lagrangian of the particle with spin maximum 1 and charge q, that is, let us perform upon the Lagrangian of Pauli the substitution (1.19) and (1.20) described above, at the same time that a wave function with $2 \times 2 = 4$ components takes the place of the wave function with two components. We thus get

$$\mathscr{L}_n{}^1 = \frac{\hbar^2}{2m_0} \nabla \chi_1^* . \nabla \chi_1 + \frac{i\hbar}{2}(\chi_1^* \partial_t \chi_1 - \chi_1 \partial_t \chi_1^*) + \frac{\hbar q}{2m_0 c}$$
$$\times \chi_1^* \frac{(\boldsymbol{\sigma} \times I) + (I \times \boldsymbol{\sigma})}{2} \chi_1 + \frac{i\hbar q}{2m_0 c}$$
$$\times \mathbf{A}.(\nabla \chi_1^* \cdot \chi_1 - \chi_1^* . \nabla \chi_1) + \left(qV + \frac{q^2}{2m_0 c^2} \mathbf{A}^2\right) \chi_1^* \chi_1$$

and one can easily make sure that the four Lagrange equations arising from $\mathscr{L}_n{}^1$ actually coincide with the equations of evolution (1.18a). Now in spite of the fact that the expression of $\mathscr{L}_n$ quite differs from that of $\mathscr{L}_n{}^1$, it can be shown that the twelve Langrange equations deriving from $\mathscr{L}_n$ are equivalent to those arising from $\mathscr{L}_n{}^1$. As a matter of fact, we get from $\mathscr{L}_n$

$$\frac{\partial \mathscr{L}_n}{\partial(\partial_k \chi_1^*)} = \frac{i\hbar c}{4}(I \times \sigma_k)\chi_2 + \frac{i\hbar c}{4}(\sigma_k \times I)\chi_3, \qquad \frac{\partial \mathscr{L}_n}{\partial(\partial_t \chi_1^*)} = \frac{i\hbar}{2}\chi_1$$

$$\frac{\partial \mathscr{L}_n}{\partial \chi_1^*} = -\frac{i\hbar}{2}\partial_t \chi_1 - qV\chi_1 - \frac{i\hbar c}{4}(I \times \boldsymbol{\sigma}).\nabla \chi_2 - \frac{i\hbar c}{4}(\boldsymbol{\sigma} \times I).\nabla \chi_3$$
$$+ \frac{q}{2}\mathbf{A}.[(I \times \boldsymbol{\sigma})\chi_2 + (\boldsymbol{\sigma} \times I)\chi_3]$$

$$\frac{\partial \mathscr{L}_n}{\partial(\partial_k \chi_2^*)} = \frac{i\hbar c}{4}(I \times \sigma_k)\chi_1, \qquad \frac{\partial \mathscr{L}_n}{\partial(\partial_t \chi_2^*)} = 0$$

$$\frac{\partial \mathscr{L}_n}{\partial \chi_2^*} = m_0 c^2 \chi_2 - \frac{i\hbar c}{4}(I \times \boldsymbol{\sigma}).\nabla \chi_1 + \tfrac{q}{2}\mathbf{A}.(I \times \boldsymbol{\sigma})\chi_1$$

$$\frac{\partial \mathscr{L}_n}{\partial(\partial_k \chi_3^*)} = \frac{i\hbar c}{4}(\sigma_k \times I)\chi_1, \qquad \frac{\partial \mathscr{L}_n}{\partial(\partial_t \chi_3^*)} = 0$$

$$\frac{\partial \mathscr{L}_n}{\partial \chi_3^*} = m_0 c^2 \chi_3 - \frac{i\hbar c}{4}(\boldsymbol{\sigma} \times I).\nabla \chi_1 + \tfrac{q}{2}\mathbf{A}.(\boldsymbol{\sigma} \times I)\chi_1,$$

so that the Lagrange equations take the form

$$\frac{i\hbar c}{2}(I \times \boldsymbol{\sigma}).\nabla\chi_2 + \frac{i\hbar c}{2}(\boldsymbol{\sigma} \times I).\nabla\chi_3$$

$$= -i\hbar\,\partial_t\chi_1 - qV\chi_1 + \frac{q}{2}\mathbf{A}.[(I \times \boldsymbol{\sigma})\chi_2 + (\boldsymbol{\sigma} \times I)\chi_3] \qquad (2.8a)$$

$$\chi_2 = \frac{i\hbar}{2m_0 c}(I \times \boldsymbol{\sigma}).\nabla\chi_1 - \frac{q}{2m_0 c^2}\mathbf{A}.(I \times \boldsymbol{\sigma})\chi_1 \qquad (2.8b)$$

$$\chi_3 = \frac{i\hbar}{2m_0 c}(\boldsymbol{\sigma} \times I).\nabla\chi_1 - \frac{1}{2m_0 c^2}\mathbf{A}.(\boldsymbol{\sigma} \times I)\chi_1 \qquad (2.8c)$$

We thus obtain equations (1.18b) and (1.18c) and by introducing (2.8b) and (2.8c) in (2.8a) we are led to

$$\frac{1}{4m_0}\left(I \times \boldsymbol{\sigma}.\left(i\hbar\nabla - \frac{q}{c}\mathbf{A}\right)\right)^2\chi_1 + \frac{1}{4m_0}\left[\boldsymbol{\sigma}.\left(i\hbar\nabla - \frac{q}{c}\mathbf{A}\right) \times I\right]^2\chi_1$$

$$= -i\hbar\,\partial_t\chi_1 - qV\chi_1$$

that is,

$$\left[\frac{1}{2m_0}\left(i\hbar\nabla - \frac{q}{c}\mathbf{A}\right)^2 + \frac{\hbar q}{2m_0 c}\frac{(I \times \boldsymbol{\sigma}) + (\boldsymbol{\sigma} \times I)}{2}.\mathbf{H}\right]\chi_1 = -i\hbar\,\partial_t\chi_1 - qV\chi_1$$

which are precisely the equations (1.8a).

Acknowledgements

We are glad to take this opportunity of thanking Professor de Broglie for his interest and valuable suggestions on this work. We are especially grateful to M. J. Andrade e Silva, to whom we owe the main subject of this work, unfailing help and useful discussions about it.

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